Implicit differentiation

Implicit differentiation: formula derivation and requirements

In many cases in mathematics and economics, the relationships between variables are presented implicitly, that is, by an equation of the form

$$F(x, y) = 0$$

where y is defined as a function of x implicitly (i.e., $y = f(x)$). To apply implicit differentiation, certain conditions must be satisfied.

Requirements for applying implicit differentiation

1. Solution verification: it is assumed that the point of interest (x_0, y_0) satisfies the equation, that is, $F(x_0, y_0) = 0$ (which implies that $y_0 = f(x_0)$)

2. Differentiability of $F(x, y)$: the function F must be differentiable in a neighborhood of the point (x_0, y_0). This ensures the existence of the partial derivatives F_x and F_y

3. Non-vanishing of the partial derivative with respect to y: it is required that $F_y(x_0, y_0) \neq 0$. This condition is essential to apply the Implicit Function Theorem, which guarantees that around (x_0, y_0), y can be expressed as a differentiable function of x (i.e., $y = f(x)$)

Derivation of the implicit derivative formula

Consider the equation

$$F(x, y) = 0$$

where we assume that $y = f(x)$ and that F is differentiable in a neighborhood of (x_0, y_0). Since $F(x, f(x)) = 0$ for all x in that neighborhood, we differentiate both sides with respect to x, applying the chain rule:

$$\frac{d}{dx}F(x, f(x)) = F_x(x, f(x)) + F_y(x, f(x)) \cdot f'(x) = 0$$

Here, F_x and F_y denote the partial derivatives of F with respect to x and y, respectively. We solve for $f'(x)$ as follows:

$$F_y(x, f(x)) \cdot f'(x) = -F_x(x, f(x))$$

$$\implies \quad f'(x) = -\frac{F_x(x, f(x))}{F_y(x, f(x))}$$

This is the general formula for the derivative of a function defined implicitly.

Example

Consider the equation of a circle:

$$x^2 + y^2 = 25$$

We define the function $F(x, y)$ as:

$$F(x, y) = x^2 + y^2 - 25 = 0$$

We observe that for any point (x, y) on the circle, $F(x, y) = 0$ holds.

Step 1: compute partial derivatives We have:

$$F_x(x, y) = 2x \quad \text{and} \quad F_y(x, y) = 2y$$

Step 2: apply the implicit derivative formula Using the formula:

$$\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)} = -\frac{2x}{2y}$$

$$\implies \quad \frac{dy}{dx} = -\frac{x}{y}$$

Implicit differentiation: function of two independent variables

In some problems in mathematics and economics, relationships between variables are given implicitly through an equation of the form

$$F(x, y, z) = 0$$

where z is implicitly defined as a function of x and y (i.e., $z = f(x, y)$). To apply implicit differentiation in this context, certain conditions must be satisfied.

Requirements for applying implicit differentiation

1. Verification of the solution: it is assumed that the point of interest $Q_0 = (x_0, y_0, z_0)$ satisfies the equation, i.e., $F(x_0, y_0, z_0) = 0$ (which implies that $z_0 = f(x_0, y_0)$)
2. Differentiability of $F(x, y, z)$: the function F and its partial derivatives F_x, F_y, and F_z must exist and be continuous in a neighborhood of the point Q_0
3. Non-vanishing of the partial derivative with respect to z: it is required that $F_z(x_0, y_0, z_0) \neq 0$. This condition is essential to express z as a differentiable function of x and y around Q_0

Derivation of the partial derivative formulas

Consider the equation

$$F(x, y, z) = 0$$

where we assume $z = f(x, y)$, and F is differentiable in a neighborhood of the point $Q_0 = (x_0, y_0, z_0)$. Since $F(x, y, f(x, y)) = 0$ for all (x, y) in that neighborhood, we differentiate both sides with respect to x and y, using the chain rule.

Application of the chain rule

To clarify the derivation, we can rewrite the function $F(x, y, z)$ by setting $u = x$, $v = y$, and $z = f(x, y)$, so that

$$w = F(u, v, z) = 0$$

Applying the chain rule with respect to x:

$$\frac{\partial w}{\partial x} = \frac{\partial F}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial F}{\partial v} \cdot \frac{\partial v}{\partial x} + \frac{\partial F}{\partial z} \cdot \frac{\partial z}{\partial x} = 0$$

Since $\frac{\partial u}{\partial x} = 1$ and $\frac{\partial v}{\partial x} = 0$, we get:

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} \cdot \frac{\partial z}{\partial x} = 0 \quad \implies \quad \frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$$

The same procedure applies to compute $\frac{\partial z}{\partial y}$

These formulas allow us to compute the partial derivatives of $z = f(x, y)$ implicitly, as long as the stated requirements are met.

Example

Consider the function:

$$F(x, y, z) = x^2y + \sin(z) - z\cos(y) + z^2 - 1 = 0$$

We observe that for any point (x, y, z) satisfying this equation, z is implicitly defined as a function of x and y (i.e., $z = f(x, y)$)

Step 1: compute partial derivatives We have:

$$F_x(x, y, z) = 2xy$$

$$F_y(x, y, z) = x^2 + z\sin(y)$$

$$F_z(x, y, z) = \cos(z) - \cos(y) + 2z$$

Step 2: apply the implicit derivative formula The formulas for the partial derivatives of $z = f(x, y)$ are:

$$\frac{\partial z}{\partial x} = -\frac{F_x(x, y, z)}{F_z(x, y, z)} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y(x, y, z)}{F_z(x, y, z)}$$

Step 3: evaluate at a point Let us choose the point $Q_0 = (1, 1, 0)$. We verify:

$$1^2 \cdot 1 + \sin(0) - 0 \cdot \cos(1) + 0^2 - 1 = 1 - 1 = 0$$

so Q_0 lies on the surface defined by $F(x, y, z) = 0$

Now we evaluate the partial derivatives at Q_0:

$$F_x(1, 1, 0) = 2 \cdot 1 \cdot 1 = 2$$

$$F_y(1, 1, 0) = 1^2 + 0 \cdot \sin(1) = 1$$

$$F_z(1, 1, 0) = \cos(0) - \cos(1) + 2 \cdot 0 = 1 - \cos(1) \approx 0.46$$

Therefore:

$$\left.\frac{\partial z}{\partial x}\right|_{(1,1,0)} = -\frac{2}{1 - \cos(1)} \approx -4.35$$

$$\left.\frac{\partial z}{\partial y}\right|_{(1,1,0)} = -\frac{1}{1 - \cos(1)} \approx -2.17$$

Second-order implicit derivatives

Consider the implicit function:

$$F(x, y, z) = 2\sin(z) - xz + y^3 - 1 = 0$$

which defines $z = z(x, y)$. We aim to compute the second-order derivatives:

$$z_{xx}, \quad z_{xy}, \quad z_{yx}, \quad z_{yy}$$

evaluated at the point $Q_0 = (1, 1, 0)$

Step 1: first-order derivatives

Since $F(x, y, z(x, y)) = 0$, we differentiate implicitly with respect to x:

$$F_x + F_z \cdot z_x = 0 \quad \Rightarrow \quad z_x = -\frac{F_x}{F_z}$$

Similarly, differentiating with respect to y:

$$F_y + F_z \cdot z_y = 0 \quad \Rightarrow \quad z_y = -\frac{F_y}{F_z}$$

We now compute the necessary partial derivatives:

$$F_x = \frac{\partial F}{\partial x} = -z, \quad F_y = \frac{\partial F}{\partial y} = 3y^2, \quad F_z = \frac{\partial F}{\partial z} = 2\cos(z) - x$$

Then:

$$z_x = \frac{z}{2\cos(z) - x}, \quad z_y = -\frac{3y^2}{2\cos(z) - x}$$

From these expressions, we apply the quotient rule to obtain the second-order derivatives, postponing numerical evaluation until the end.

Computation of z_{xx}

Differentiate $z_x = \frac{z}{2\cos(z)-x}$ with respect to x:

$$z_{xx} = \frac{d}{dx}\left(\frac{z}{2\cos(z) - x}\right) = \frac{z_x(2\cos(z) - x) - z \cdot (-2\sin(z)z_x - 1)}{(2\cos(z) - x)^2}$$

$$z_{xx} = \frac{z_x(2\cos(z) - x) + z(2\sin(z)z_x + 1)}{(2\cos(z) - x)^2}$$

Evaluating at $Q_0 = (1, 1, 0)$ (where $z = 0, z_x = 0, \sin(0) = 0, \cos(0) = 1, x = 1$):

$$z_{xx}(1, 1) = \frac{(0)(2(1) - 1) + 0(2(0)(0) + 1)}{(2(1) - 1)^2} = \frac{0 + 0}{1^2} = 0$$

Computation of z_{xy}

Differentiate $z_x = \frac{z}{2\cos(z)-x}$ with respect to y:

$$z_{xy} = \frac{d}{dy}\left(\frac{z}{2\cos(z) - x}\right) = \frac{z_y(2\cos(z) - x) - z \cdot (-2\sin(z)z_y)}{(2\cos(z) - x)^2}$$

$$z_{xy} = \frac{z_y(2\cos(z) - x) + 2z\sin(z)z_y}{(2\cos(z) - x)^2}$$

Evaluating at $Q_0 = (1, 1, 0)$ (with $z = 0, z_y = -3, \sin(0) = 0, \cos(0) = 1, x = 1$):

$$z_{xy}(1, 1) = \frac{(-3)(2(1) - 1) + 2(0)(0)(-3)}{(2(1) - 1)^2} = \frac{-3 + 0}{1^2} = -3$$

By symmetry of mixed partial derivatives:

$$z_{yx} = z_{xy}$$

Computation of z_{yy}

Differentiate $z_y = -\frac{3y^2}{2\cos(z)-x}$ with respect to y:

$$z_{yy} = -\frac{(6y)(2\cos(z)-x) + (3y^2)(-2\sin(z)z_y)}{(2\cos(z)-x)^2}$$

$$z_{yy} = -\frac{6y(2\cos(z)-x) + 6y^2\sin(z)z_y}{(2\cos(z)-x)^2}$$

Evaluating at $Q_0 = (1,1,0)$ (with $y = 1, z = 0, z_y = -3, \sin(0) = 0, \cos(0) = 1, x = 1$):

$$z_{yy}(1,1) = -\frac{6(1)(2(1)-1) + 6(1)^2(0)(-3)}{(2(1)-1)^2} = -\frac{6(1)+0}{1^2} = -6$$